Termination of Isabelle Functions via Termination of Rewriting

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What?

Why?

How?

What?

Why?

How?

What?

Why?

How?

talk

What? talk

Why?

How?

paper

Functional Programming in Isabelle/HOL

Datatypes

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datatype tree =
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| N tree nat tree (*node having 2 subtrees*)
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(*assuming search tree property*)
fun getmax :: "tree => nat" where
  "getmax E            =  0"
| "getmax (N _ x E)   =  x"
| "getmax (N _ _ r)   =  getmax r"
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Consider

```
fun f where "f x = f x + 1"
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Proving Totality of Isabelle/HOL Functions

Built-In Automation

- primitive recursion (syntactic)
- lexicographic orders
- size-change principle

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- use external (fully automatic) termination tool

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First-Order Term Rewriting - "Replacing Equals by Equals"

Definition by Example

$$\begin{aligned} \mathsf{getmax}(\mathsf{E}) &\to \mathsf{0} \\ \mathsf{getmax}(\mathsf{N}(\mathsf{x}, \mathsf{y}, \mathsf{E})) &\to \mathsf{y} \\ \mathsf{getmax}(\mathsf{N}(\mathsf{x}, \mathsf{y}, \mathsf{N}(\mathsf{z}, \mathsf{u}, \mathsf{v}))) &\to \mathsf{getmax}(\mathsf{N}(\mathsf{z}, \mathsf{u}, \mathsf{v})) \end{aligned}$$

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term rewrite system (TRS)

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Termination Techniques

transformations (semantic labeling, root-labeling, uncurrying, . . .), interpretations (polynomial, matrix, arctic, . . .), orders (Knuth-Bendix, lexicographic, multiset, RPO, . . .), advanced methods (dependency pairs, dependency graph, usable rules, . . .), . . .

First-Order Term Rewriting

Termination Tools

AProVE, C*i*ME, Jambox, Matchbox, NTI, VMTL, Torpa, TPA, $\mathsf{T_TT_2}$, ...

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Problems

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- not stable (introduction of new techniques)
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Solutions

- XML format for proofs (Certification Problem Format - CPF)
- automatic certification of CPF files (using a proof assistant)

Two Worlds

Totality of Isabelle/HOL Functions

- input: defining equations E_f for function f of type `'a => 'b`
- output: call-relation $\mathcal{C}_f$ of type `('a × 'a) set`
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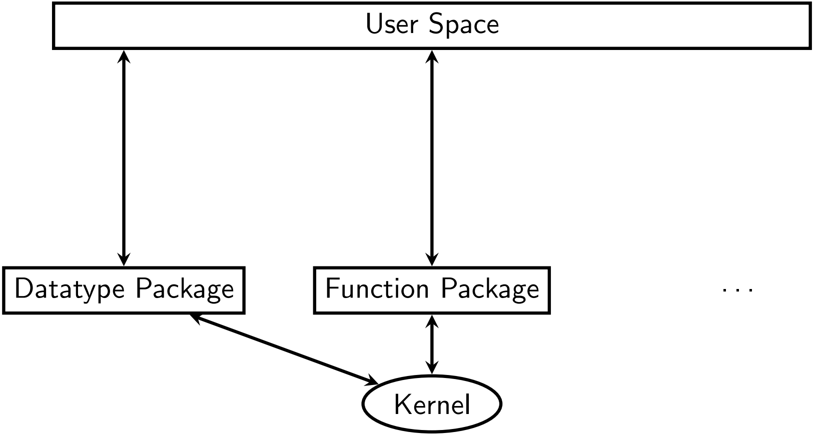
The Big Picture



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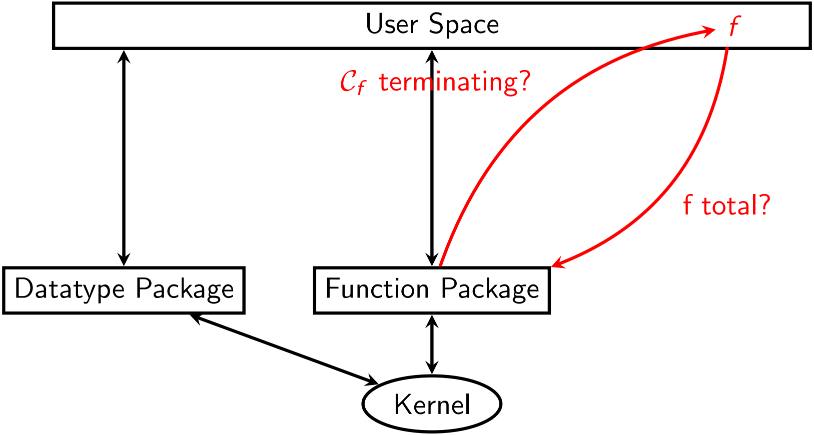

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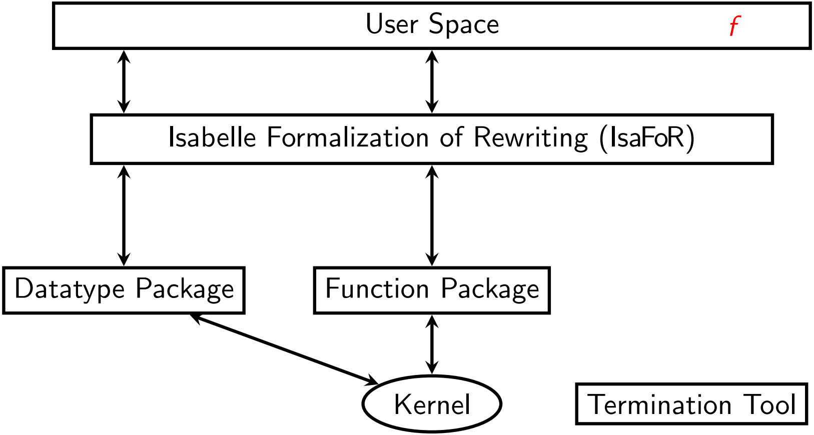

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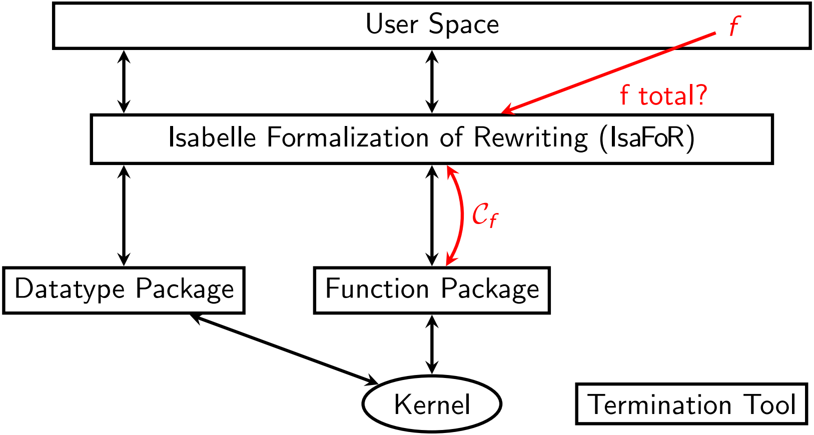

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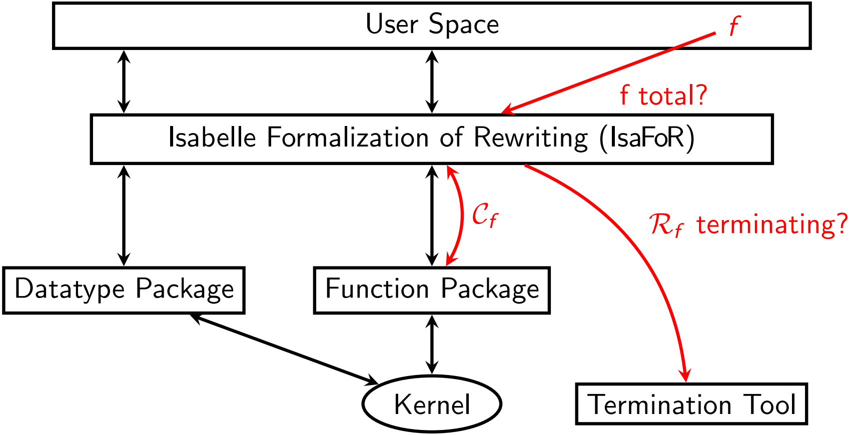

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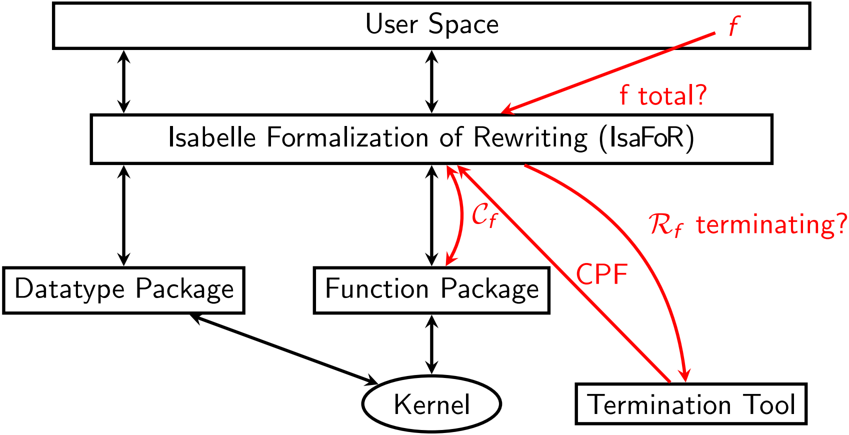

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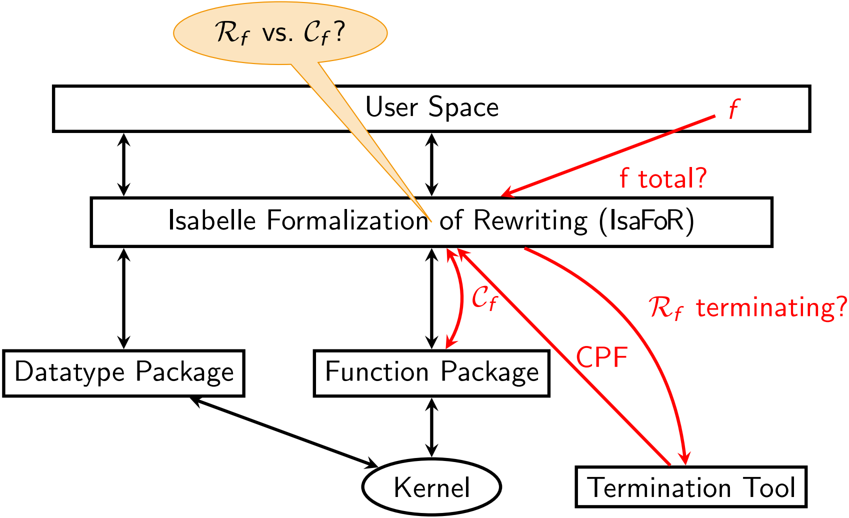

Necessary Glue

- import CPF certificate into Isabelle (using IsaFoR)
- generate TRS $\mathcal{R}_f$ corresponding to definition of function f
- relate termination of $\rightarrow_{\mathcal{R}_f}$ to well-foundedness of $\mathcal{C}_f$?

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- internal type

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term = Var string | Fun string (term list)
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- rewrite rules for equations $l_1 = r_1, \ldots, l_k = r_k$

$$\textsc{rules}(f) = \{ \begin{array}{l} \textsc{enc}(l_1) \rightarrow \textsc{enc}(r_1), \\ \vdots \\ \textsc{enc}(l_k) \rightarrow \textsc{enc}(r_k)\} \end{array}$$

Main Goal

- encoding *emb* of type `'a => term`
- prove that $\mathcal{C}_f$ is contained in

$$\{(x, y) \mid \texttt{Fun f [}\mathit{emb}\ x\texttt{]}\ (\rightarrow_{\mathcal{R}_f} \cup \rhd)^+\ \texttt{Fun f [}\mathit{emb}\ y\texttt{]}\}$$

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Simulation Lemmas

- n-ary function f
- lemma:

$$\texttt{Fun f [}\mathit{emb}\ x_1, \ldots, \mathit{emb}\ x_n\texttt{]} \rightarrow^*_{\mathcal{R}_f} \mathit{emb}\ (f\ \vec{x}_n)$$

Restrictions

Supported

- variables, function applications
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Not Supported

- no data type constructors with functional arguments
- no “lambdas”
- no function variables
- no overlapping patterns
- no incomplete patterns
- no mutual recursion

Summary

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prove termination of Isabelle/HOL functions by external termination tool

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How

refer to paper and Isabelle/HOL formalization
`http://cl-informatik.uibk.ac.at/software/ceta`