Three Chapters of Measure Theory in Isabelle/HOL

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ITP 2011
23 August 2011

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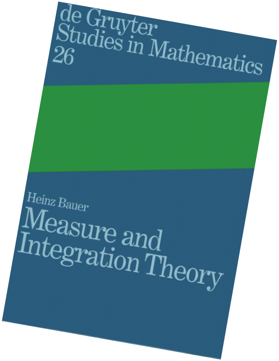

Heinz Bauer: Measure and Integration Theory

(Most of the first three chapters)

Motivation: Program Analysis

- Randomness sources: input, scheduling, random number generator, ...
- Measure sets of sequences of random numbers, scheduling decisions, ...
  ($\mathbb{N} \Rightarrow \alpha$, non-countable spaces)

Concepts

Existence	Show existence of the measure spaces of traces
Lebesgue integral	Expected output size, ...
Density	Information flowing from input to output (Information theory)
Product	Combine different sources in one measure space

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σ-Algebra

$$
\begin{aligned}
&\texttt{record}\ \alpha\ \mathit{set\text{-}system} = \\
&\qquad \mathit{space} \quad :: \quad \alpha\ \mathit{set} \qquad \text{“}\Omega\text{”} \\
&\qquad \mathit{sets} \quad :: \quad \alpha\ \mathit{set}\ \mathit{set} \qquad \text{“}\mathscr{A}\text{”}
\end{aligned}
$$

σ-Algebra

```
record α set-system =
    space :: α set       "Ω"
    sets  :: α set set   "𝒜"
```

- Set systems are introduced as locales

locale *algebra*

$\mathscr{A} \subseteq \mathscr{P}(\Omega)$,
$\emptyset, \Omega / A$,
$A \cup B \in \mathscr{A}$

↑

locale *σ-algebra*

$\bigcup_{\mathbb{N}} F \in \mathscr{A}$

⋮

σ-Algebra

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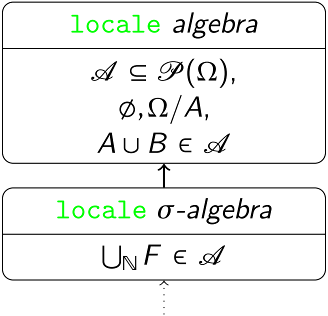
- $\sigma(\mathscr{G})$:= smallest σ-algebra containing $\mathscr{G}$



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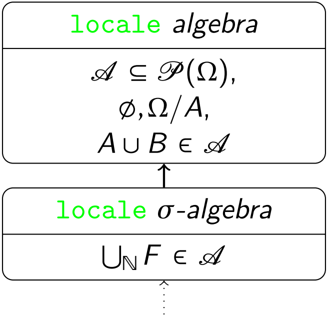
- Example: Traces $\sigma(\{\{t | t \textit{ starts with } ts\} | ts\})$



Measure Spaces

datatype $\overline{\mathbb{R}} = -\infty \mid$ **ereal** $\mathbb{R} \mid \infty$

record α *measure-space* =
  α *set-system* +
  measure :: α *set* $\Rightarrow \overline{\mathbb{R}}$ "μ"

locale σ-*algebra*

↑

locale *measure-space*

$\mu(\emptyset) = 0 \leq \mu(A)$,
disj $F \rightarrow \mu(\bigcup F) = \sum_i \mu(F_i)$

↑

locale σ-*finite-measure-space*

$\exists F.\ \bigcup_{\mathbb{N}} F = \Omega \ \wedge\ \mu(F_i) < \infty$

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locale *finite-measure-space*

$\mu(\Omega) < \infty$

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Caratheodory: Extends a measure on an algebra $\mathscr{G}$ to its σ-algebra $\sigma(\mathscr{G})$

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Uniqueness σ-finite measures equal on an $\cap$-stable set $\mathcal{G}$ are equal on its σ-algebra $\sigma(\mathcal{G})$

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Lebesgue Integral

$$\begin{aligned}
\int^{+} \cdot \quad &:: \quad (\alpha \Rightarrow \overline{\mathbb{R}}) \Rightarrow \overline{\mathbb{R}} \\
\int \cdot \quad &:: \quad (\alpha \Rightarrow \mathbb{R}) \Rightarrow \mathbb{R} \\
\int_x f(x) \quad &= \quad \int_x^{+} f^{+}(x) - \int_x^{+} f^{-}(x) \\
\text{integrable } f \quad &\longleftrightarrow \quad \text{measurable } f \ \wedge \\
&\qquad \int_x^{+} f^{+}(x) < \infty \ \wedge \ \int_x^{+} f^{-}(x) < \infty
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Linearity (f, g measurable): $\int^{+}(c \cdot f + d \cdot g) = c \cdot \int^{+} f + d \cdot \int^{+} g$

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Density := The measurable function f to represent ν as the integral over μ:

$$\forall A. \qquad \nu(A) = \int_X^+ f(x) \cdot \chi_A(x) d\mu$$

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Uniqueness Two densities from μ to ν are almost everywhere equal (σ-finite).

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Product Measure

Sets := $\mathscr{A} \otimes \mathscr{B}$ is the σ-closure of all $A \times B$

Measure := $\mu \otimes \nu$ (σ-finite μ and ν) on $\mathscr{A} \otimes \mathscr{B}$ with:

$$\mu \otimes \nu(A \times B) = \mu(A) \cdot \nu(B)$$

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Fubini
$$\int_{(x,y)} f(x,y) d(\mu \otimes \nu) = \int_x \left(\int_y f(x,y) d\nu \right) d\mu = \int_y (\int_x f(x,y) d\mu) d\nu$$

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Symmetry Symmetric variants are proven by the isomorphism swap $(\lambda(x,y).(y,x))$

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Iterated Product: Over a finite I: $\bigotimes_{i \in I} \mathscr{A}_i :: (\iota \Rightarrow \alpha)$ *measure-space*

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But measure theory is even more ...

Multivariate Analysis

- $\sigma\left(\left\{\,[a,b[\;\middle|\,a,b\in\mathbb{R}\right\}\right)$ – called Borel sets
- On euclidean spaces equivalent to $\sigma(\{X|\text{ open }X\})$
  $\implies$ define Borel sets on *topological-space*s ($\mathbb{R}$, $\mathbb{R}^n$, $\overline{\mathbb{R}}$, ...)

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- Existence by Gauge integral over χ_A, *not Caratheodory*
- Euclidean space equal to products:
  $\bigotimes_{i<n} \lambda \approx \lambda^n$
- Lebesgue integral subset of Gauge integral:

$$\begin{aligned} \int_X^G f(x) &= \int_X f(x) d\lambda^n \\ &= \int_X f(x) d\left(\bigotimes_{i<n} \lambda\right) \end{aligned}$$

Related Work

	HOL							
	Hurd	Richter	Coble	Mhamdi	Lester (PVS)	Mizar	HOL-Light	Isabelle
$\overline{\mathbb{R}}$				ITP'11	✓	✓		✓
Borel Sets (Open)				✓	✓			✓
Leb. Integral		✓	✓	✓	✓	✓	✓	✓
Leb. Measure	$\mathbb{N} \Rightarrow \mathbb{B}$				✓	✓	$\lambda \cdot \in \mathbb{R}$	✓
Products			finite		✓		$\mathbb{R}^{\alpha+\beta}$	✓
Dynkin						✓		✓
Radon-Nikodým			finite	ITP'11				✓

Applications (Not in Paper)

Information Theory Use Radon-Nikodým derivative to formalize relative entropy (Kullback-Leibler divergence), Mutual Information and Entropy

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Probability Theory Existence of infinite products ($\mathbb{N} \Rightarrow [0,1[$), independence of random variables

Markov Chains Formalize measure of Markov chains traces